A UNIVERSAL APPROXIMATE FORMULA FOR CHARACTERISTIC IMPEDANCE OF STRIP TRANSMISSION LINES WITH RECTANGULAR INNER CONDUCTORS*

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Abstract

An explicit expression is developed for the characteristic impedance of a microwave strip transmission line with rectangular inner conductor of arbitrary dimensions. The expression is exact for zero thickness and arbitrary width, exact for zero width and arbitrary thickness, and quite accurate (within 3% for the extreme cases of a square inner conductor of dimensions about 0.01 of plate separation, but in most cases of practical interest within 0.1%) throughout the entire range of thickness and width.

List of Symbols

b	parameter related to thickness/separation ratio
c	arbitrary coefficient to be evaluated
f	expansion parameter $\equiv e^{-\pi\rho/\sigma}$
g	suitably weighted mean of g_L and g_H
g_H	expansion parameter in expression for Z_{0H}
g_L	expansion parameter in expression for Z_{0L}
k	modulus of elliptic function
$\bar{k}$	modulus of elliptic function
z_H	parameter associated with Z_H
z_L	parameter associated with Z_L
F	specially defined function of β
G	specially defined function of β
K	elliptic integral
K'	elliptic integral of complement
Z_0	characteristic impedance of thick line in free space
Z_{0H}	upper bound to Z_0
Z_{0L}	lower bound to Z_0
Z_{00}	characteristic impedance of zero-thickness line in free space
Z_{00H}	upper bound to Z_{00}
Z_{00L}	lower bound to Z_{00}
β	ratio of thickness of center conductor to separation of ground planes
κ_e	dielectric constant of medium between plates
v	half-thickness of center conductor
ρ	half-width of center conductor
$\bar{\rho}$	parameter related to ρ, σ, and β
σ	half-separation of ground planes

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I. Introduction

Considerable interest has been shown of late in so-called "strip" or "shielded-strip" transmission lines, consisting of an inner conductor of rectangular, or nearly rectangular, cross section placed symmetrically between two infinite grounded planes. A useful model of such a transmission line is shown in cross section in Fig. 1.

Various approaches to the calculation of the characteristic impedance of such a line have been described at this symposium. The approach forming the subject of this report consists of (1) deriving a rough formula for characteristic impedance by taking the harmonic mean of rigorous upper and lower bounds,[1] and (2) refining this expression by requiring it to be identical with the exact expressions for the two limiting cases of (a) zero thickness and arbitrary width and (b) zero width and arbitrary thickness.

The result is consequently exact for these two cases. It is extremely accurate in the low impedance region where the bounds of Reference 1 are very close together. Furthermore, since it is exact for the high-impedance limits of zero thickness and zero width, it gives a good account of the impedance throughout its entire range.

II. Derivation of Universal Formula

It was shown in a previous investigation[2] that rigorous lower and upper limits to the characteristic impedance of a strip transmission line with rectangular inner conductor may be obtained in the respective forms

$$\sqrt{\kappa_e}\, Z_{0L} = \frac{30\pi^2}{\ln \frac{z_L + 1}{z_L - 1}} \text{ ohms} \qquad (2.1a)$$

$$\sqrt{\kappa_e}\, Z_{0H} = \frac{30\pi^2}{\ln \frac{1 + z_H}{1 - z_H}} \text{ ohms} \qquad (2.1b)$$

where z_L and z_H are two parameters related to the width/separation ratio (ρ/σ) and the thickness/separation ratio

$$\frac{\gamma}{\sigma} = \beta = 1 - \sqrt{1 - b^2} \qquad (2.2)$$

through the respective equations

$$-\frac{\pi\rho}{\sigma} = \ln \frac{z_L + \sqrt{z_L^2 - b^2}}{z_L - \sqrt{z_L^2 - b^2}}$$

$$-\sqrt{1-b^2}\, \ln \frac{\sqrt{z_L^2-b^2} + z_L\sqrt{1-b^2}}{\sqrt{z_L^2-b^2} - z_L\sqrt{1-b^2}} \qquad (2.3a)$$

and

$$-\frac{\pi\rho}{\sigma} = \ln \frac{z_H + \sqrt{z_H^2-b^2}}{z_H - \sqrt{z_H^2-b^2}}$$

$$-\sqrt{1 - b^2}\, \ln \frac{z_H\sqrt{1-b^2} + \sqrt{z_H^2-b^2}}{z_H\sqrt{1-b^2} - \sqrt{z_H^2-b^2}} \qquad (2.3b)$$

From inspection of Eq. (2.1) it may be deduced that, for values of Z_0 in common use, z_L and z_H do not differ greatly from from unity. (For example, values of 1.1 and 0.9, respectively, correspond to impedances of the order of 100 **ohms**.) Consequently it is useful to express these two parameters as

$$z_L = \sqrt{1 + g_L} \quad (\text{def. of } g_L) \qquad (2.4a)$$

$$z_H = \sqrt{1 + g_H} \quad (\text{def. of } g_H) \qquad (2.4b)$$

and expand in powers of g_L and g_H; substitutions of (2.4a) into (2.1a), and (2.4b) into (2.1b), yield the respective series

$$Z_{0L} = \frac{30\pi^2}{\ln \frac{4}{g_L} + \frac{g_L}{2} - \frac{3}{16}g_L^2 + \cdots} \qquad (2.5a)$$

and

$$Z_{0H} = \frac{30\pi^2}{\ln \frac{4}{g_H} - \frac{g_H}{2} - \frac{3}{16}g_H^2 - \cdots} \qquad (2.5b)$$

Valuable information as to the direction in which it is most fruitful to proceed may be obtained from the corresponding expressions for the case of zero thickness;[3] there it is found that the lower and upper limits to, and the exact expression for, characteristic impedance may be written in series form respectively as

$$\sqrt{\kappa_e}\, Z_{00L} = \frac{30\pi^2}{\ln\frac{4}{f} + \frac{f}{2} - \frac{3}{16}f^2 + \cdots} \qquad (2.6a)$$

$$\sqrt{\kappa_e}\, Z_{00H} = \frac{30\pi^2}{\ln\frac{4}{f} - \frac{f}{2} - \frac{3}{16}f^2 - \cdots} \qquad (2.6b)$$

$$\sqrt{\kappa_e}\, Z_{00} = 30\pi\frac{K'(k)}{K(k)} \qquad (2.6c)$$

$$= \frac{30\pi^2}{\ln\frac{4}{f} - \frac{3}{16}f^2 + \cdots} \qquad (2.6d)$$

$$= \frac{30\pi^2}{\ln 4 + \frac{\pi\rho}{\sigma} - \frac{3}{16}e^{-2\pi\rho/\sigma} + \cdots} \qquad (2.6e)$$

where

$$k = \tanh\frac{\pi\rho}{2\sigma} \qquad (2.6f)$$

$$f \equiv e^{-\pi\rho/\sigma} \qquad (2.6g)$$

In the above zero-thickness case, the exact value of impedance is rather closely the harmonic mean of the bounds; accordingly, a similar assumption will be made for the finite thickness case, and the characteristic impedance will be tentatively expressed in the form

$$\sqrt{\kappa_e}\, Z_0 \simeq \frac{2}{\frac{1}{\sqrt{\kappa_e}Z_{0L}} + \frac{1}{\sqrt{\kappa_e}Z_{0H}}}$$

$$\simeq \frac{30\pi^2}{\frac{1}{2}\left(\ln\frac{4}{g_L} + \ln\frac{4}{g_H}\right) + (\text{term in } g^2) + \text{higher terms}} \qquad (2.7)$$

where g is a quantity of the order of g_L and g_H.

Now Eqs. (2.3a) and (2.3b) may be re-expressed respectively as

$$-\frac{\pi\rho}{\sigma} = \ln\left[\left(2 - \frac{b^2}{z_L^2} + 2\sqrt{1 - \frac{b^2}{z_L^2}}\right)\frac{z_L^2}{b^2}\right]$$

$$-\sqrt{1-b^2}\,\ln\left\{\left[2 - b^2 - \frac{b^2}{z_L^2} + 2\sqrt{(1-b^2)\left(1 - \frac{b^2}{z_L^2}\right)}\right]\frac{z_L^2}{b^2}\right\}$$

$$-\sqrt{1-b^2}\,\ln\frac{1}{z_L^2-1} \qquad (2.8a)$$

and

$$-\frac{\pi\rho}{\sigma} = \ln[(2 - \frac{b^2}{z_H^2} + 2\sqrt{1 - \frac{b^2}{z_H^2}})\frac{z_H^2}{b^2}]$$

$$-\sqrt{1-b^2}\ln\left\{[2-b^2-\frac{b^2}{z_H^2}+2\sqrt{(1-b^2)(1-\frac{b^2}{z_H^2})}]\frac{z_H^2}{b^2}\right\}$$

$$-\sqrt{1-b^2}\ln\frac{1}{1-z_H^2}; \qquad (2.8b)$$

then, using (2.4),

$$\sqrt{1-b^2}\ln\frac{1}{g_L} = \frac{\pi\rho}{\sigma} + \ln[\frac{2(1+g_L)}{b^2}(1+\sqrt{1-\frac{b^2}{1+g_L}})-1]$$

$$-\sqrt{1-b^2}\ln\left\{\frac{2(1+g_L)}{b^2}[1+\sqrt{(1-b^2)(1-\frac{b^2}{1+g_L})}]\right.$$

$$\left.-(1+g_L)-1\right\} \qquad (2.9a)$$

and

$$\sqrt{1-b^2}\ln\frac{1}{g_H} = \frac{\pi\rho}{\sigma}+\ln[\frac{2(1-g_H)}{b^2}(1+\sqrt{1-\frac{b^2}{1-g_H}}) - 1]$$

$$-\sqrt{1-b^2}\ln\left\{\frac{2(1-g_H)}{b^2}[1+\sqrt{(1-b^2)(1-\frac{b^2}{1-g_H})}]\right.$$

$$\left.-(1-g_H)-1\right\} \qquad (2.9b)$$

A first approximation to the leading term in the denominator of the right-hand side of Eq. (2.7) may be obtained by setting $g_L = 0$ on the right-hand side of (2.9a), or $g_H = 0$ on the right-hand side of (2.9b), i.e.,

$$\frac{\sqrt{1-b^2}}{2}(\ln\frac{1}{g_L}+\ln\frac{1}{g_H}) = \frac{\pi\rho}{\sigma}+\ln[\frac{2}{b^2}(1+\sqrt{1-b^2}) - 1]$$

$$-\sqrt{1-b^2}\ln\left\{\frac{2}{b^2}[1+(1-b^2)] - 2\right\} \qquad (2.10)$$

+ terms of order g^2 + higher terms;

the additional terms are of the order of g^2 and above rather than of the order of g due to first-order cancellations of the type

$$\ln(1+g) + \ln(1-g) = (g-\frac{g^2}{2}+\dots)$$

$$+(-g-\frac{g^2}{2}-\dots) = -g^2 + \text{higher terms}.$$

With the help of Eq. (2.2), (2.10) may be written in the more useful form

$$\frac{1}{2}(\ln\frac{1}{g_L}+\ln\frac{1}{g_H}) = \frac{1}{1-\beta}[\frac{\pi\rho}{\sigma}+\beta\ln\frac{2}{\beta}+2(1-\beta)\ln\frac{1}{1-\beta}$$

$$-2(1-\frac{\beta}{2})\ln\frac{1}{1-\frac{\beta}{2}}] \qquad (2.11)$$

+ terms of order g^2 + higher terms, whence, if (2.1) be substituted into (2.7), there results a first approximate formula for characteristic impedance:

$$\sqrt{\kappa_e}Z_o = \frac{30\pi^2}{\ln 4+\frac{1}{1-\beta}[\frac{\pi\rho}{\sigma}+\beta\ln\frac{2}{\beta}+2(1-\beta)\ln\frac{1}{1-\beta}-2(1-\frac{\beta}{2})\ln\frac{1}{1-\frac{\beta}{2}}] + \text{terms of order } g^2 + \text{higher terms}} \qquad (2.12)$$

If terms of order g^2 and higher are neglected in (2.12), the result of J. J. Thomson[4] is obtained, in which end corrections for an infinite width line only are applied.

The first refinement of (2.12) is a re-expression in terms of elliptic functions, (with a corresponding change in the magnitude of the terms in g^2 and above) such that Z_o is exact for zero thickness ($\beta = 0$). Let an effective width $\bar{\rho}$ be approximately defined through

$$\frac{\pi\bar{\rho}}{\sigma} = \frac{1}{1-\beta}[\frac{\pi\rho}{\sigma}+\beta\ln\frac{2}{\beta}+2(1-\beta)\ln\frac{1}{1-\beta} -2(1-\frac{\beta}{2})\ln\frac{1}{1-\frac{\beta}{2}}]$$

+ terms in g^2 to be determined later
+ higher terms (2.13)

From (2.10) it follows that, if g is considered a suitably weighted average of g_L and g_H,

$$g \doteq e^{-\pi\bar{\rho}/\sigma} + \text{higher terms} \qquad (2.14)$$

whence, substituting (2.13) and (2.14) into (2.12),

$$\sqrt{\kappa_e}Z_o \doteq \frac{30\pi^2}{\ln 4+\frac{\pi\bar{\rho}}{\sigma}+ \text{terms in } e^{-2\pi\bar{\rho}/\sigma} + \text{higher terms}} \qquad (2.15)$$

Then comparison of (2.15) with (2.6e) shows that, if the coefficient of $e^{-2\pi\bar{\rho}/\sigma}$ in (2.15) is arbitrarily set at (-3/16), an expression for characteristic impedance which is exact for $\beta = 0$ is

$$\sqrt{\kappa_e}Z_o \doteq 30\pi\frac{K'(\bar{k})}{K(\bar{k})} \qquad \bar{k} \doteq \tanh\frac{\pi\bar{\rho}}{2\sigma} \qquad (2.16)$$

As yet the coefficient of g^2 in (2.13) is unspecified; the first refinement did not fix it directly. The second refinement of the characteristic impedance formula consists in setting the coefficient such that Z_o be exact for zero width ($\rho = 0$). For $\rho = 0$, the value of $\bar{k}$ in Eq. (2.16) must be given by[5]

$$\bar{k} = \sin\frac{\pi v}{2} = \sin\frac{\pi\beta}{2} \qquad (\rho = 0). \qquad (2.17)$$

Consequently, if (2.13) be re-written as the definition

$$\frac{\pi\bar{\rho}}{\sigma} = \frac{1}{1-\beta}\left[\frac{\pi\rho}{\sigma}+\beta\ln\frac{2}{\beta}+2(1-\beta)\ln\frac{1}{1-\beta} - 2(1-\frac{\beta}{2})\ln\frac{1}{1-\frac{\beta}{2}}\right] + ce^{-\frac{2}{1-\beta}\frac{\pi\rho}{\sigma}} \qquad (2.18)$$

where the ln terms in β in the exponent are absorbed into the constant and higher terms neglected, then substitution of (2.18) into condition (2.17) results in the requirement

$$\tanh^{-1}(\sin\frac{\pi\beta}{2}) = (\frac{\pi\bar{\rho}}{2\sigma})_{\rho=0}$$

$$= \frac{1}{2(1-\beta)}\left[\beta\ln\frac{2}{\beta}+2(1-\beta)\ln\frac{1}{1-\beta} - 2(1-\frac{\beta}{2})\ln\frac{1}{1-\frac{\beta}{2}}\right] + \frac{c}{2}$$

or

$$c = 2\tanh^{-1}(\sin\frac{\pi\beta}{2}) - \frac{1}{1-\beta}\left[\beta\ln\frac{2}{\beta}+2(1-\beta)\ln\frac{1}{1-\beta} - 2(1-\frac{\beta}{2})\ln\frac{1}{1-\frac{\beta}{2}}\right] \qquad (2.19)$$

and

$$\frac{\pi\rho}{\sigma} = \frac{1}{1-\beta}\frac{\pi\rho}{\sigma}+\frac{1}{1-\beta}\left[\beta\ln\frac{2}{\beta}+2(1-\beta)\ln\frac{1}{1-\beta} - 2(1-\frac{\beta}{2})\ln\frac{1}{1-\frac{\beta}{2}}\right](1-e^{-\frac{2}{1-\beta}\frac{\pi\rho}{\sigma}}) + 2\left[\tanh^{-1}(\sin\frac{\pi\beta}{2})\right]e^{-\frac{2}{1-\beta}\frac{\pi\rho}{\sigma}} \qquad (2.20)$$

Consequently, if for simplicity the functions

$$F(\beta) \equiv \frac{1}{\pi(1-\beta)}\left[\beta\ln\frac{2}{\beta}+2(1-\beta)\ln\frac{1}{1-\beta} - 2(1-\frac{\beta}{2})\ln\frac{1}{1-\frac{\beta}{2}}\right] \qquad (2.21a)$$

and

$$G(\beta) \equiv F(\beta) - \frac{2}{\pi}\tanh^{-1}(\sin\frac{\pi\beta}{2})$$

$$= F(\beta) - \frac{1}{\pi}\ln\frac{1+\sin(\pi\beta/2)}{1-\sin(\pi\beta/2)} \qquad (2.21b)$$

are defined, then it is possible to obtain readily the quantity

$$\frac{\bar{\rho}}{\sigma} \doteq \frac{1}{1-\beta}\frac{\rho}{\sigma} + F(\beta) - G(\beta)\,e^{-\frac{2}{1-\beta}\frac{\pi\rho}{\sigma}} \qquad (2.21c)$$

and then the quantity

$$\bar{k} \equiv \tanh\frac{\pi\bar{\rho}}{2\sigma} \qquad (2.21d)$$

whence the characteristic impedance is given approximately by

$$\sqrt{\kappa_e}\,Z_o \doteq 30\,\frac{K'(\bar{k})}{K(\bar{k})} \qquad (2.21e)$$

III. Tests of Universal Formula

Table I compares the values of $\sqrt{\kappa_e}Z_o$ for $(\rho/\sigma) = 0.2, 0.5, 1.0$ and $(v/\sigma) = 0.2, 0.5$ computed by the universal formula (2.21) with bounds and estimates obtained from previous analyses.[1,6] The estimates are, as far as can be predicted, accurate to at least the figures given in the table. In the region of primary physical interest the predictions of the universal formula are accurate to within about 0.1% or less.

To test (2.21) under the least favorable circumstances, three further numerical calculations were carried out for very small square inner conductors and the results compared in Table II with those of Reference 6. The accuracy of the universal formula in the region of extremely high impedance may be determined easily by assuming $\rho \ll \sigma$ and $v \ll \sigma$ in (2.21) and in the results of Reference 6. From (2.21) it follows that

$$\frac{\bar{\rho}}{\sigma} \doteq \frac{\rho}{\sigma} + F(\beta) - G(\beta) \doteq \frac{\rho+v}{\sigma}$$

$$\bar{k} \doteq \frac{\pi\bar{\rho}}{2\sigma} \doteq \frac{\pi}{2}(\frac{\rho+v}{\sigma}) \qquad \begin{pmatrix}\rho\ll\sigma\\ v\ll\sigma\end{pmatrix} \text{ (calc)} \qquad (3.1)$$

$$Z_o \doteq 30\pi\frac{\ln(4/\bar{k})}{(\pi/2)} \doteq 60\ln\frac{4}{\bar{k}} \doteq 60\ln\frac{8}{\pi}\frac{\sigma}{\rho+v},$$

while from Reference 6 it follows that

$$k \doteq \frac{\pi\rho_o}{2\sigma} \qquad \begin{pmatrix}\rho\ll\sigma\\ v\ll\sigma\end{pmatrix} \text{ (Ref.6)}$$

$$Z_o \doteq 60\ln\frac{4}{k} \doteq 60\ln\frac{8}{\pi}\frac{\sigma}{\rho_o}, \qquad (3.2)$$

whence

$$Z_o(\text{calc.}) \doteq Z_o(\text{Ref.6}) + 60\ln\frac{\rho_o}{\rho+v} \qquad \begin{pmatrix}\rho\ll\sigma\\ v\ll\sigma\end{pmatrix} \qquad (3.3)$$

The difference depends upon the ratio (v/ρ); for the extreme case of $v = \rho$, $(\rho/\rho_o) = 0.4235_5$. Hence

$$Z_o(\text{calc.}) - Z_o(\text{Ref.6}) \lesssim 60\ln\frac{1}{0.8471}$$

$$Z_o(\text{calc.}) - Z_o(\text{Ref.6}) \cong 10 \text{ ohms} \qquad (3.4)$$

The maximum possible error is 10 ohms; from inspection of Table II one may conclude that this value is reached for Z_o somewhat larger than 300 ohms. Hence the error for a square inner conductor reaches a peak of about 3% at about 300 ohms and then declines; the percentage error for inner conductors of other rectangular cross sections is less. This error is inherent in the linear nature of the expressions used and cannot be eliminated simply. However, in a region of such high impedance, Z_o may be calculated quickly and accurately by the methods of Reference 6.

IV. Conclusion

The present paper offers a single straightforward, if slightly lengthy, means of computing characteristic impedance for a rectangular center conductor of any dimensions, which is extremely accurate not only in most cases of practical interest but also in many cases of academic interest.

Furthermore, the functions $F(\beta)$ and $G(\beta)$ become much less complicated when β is small; work on a very simple but very accurate formula for thin lines of reasonable width is in progress.

References

1. R. L. Pease, Interim Report No.2 on Contract No.AF 19(604)-575, Research Laboratory of Physical Electronics, Tufts College; January 12, 1954. The method of this report was based upon a suggestion by H.A. Wheeler, to whom the author is greatly indebted.

2. Reference 1, p. 19.

3. Reference 1, pp. 15-17.

4. J.J. Thomson, "Notes on Recent Researches in Electricity and Magnetism," Oxford, 1893, Sec. 237.

5. F. Oberhettinger and W. Magnus, Anwendungen der Elliptischen Funktionen in Physik und Technik, Springer, p. 63; 1949.

6. R.L. Pease, Interim Report No.5 on Contract AF 19(604)-575, Research Laboratory of Physical Electronics, Tufts College; May 10, 1954.

Table I. Check on universal impedance formula in low and medium impedance range. ($\kappa_e = 1$)

$\frac{v}{\sigma}$ \ $\frac{\rho}{\sigma}$ =	0	0.2	0.5	1.0
0	EXACT	EXACT	EXACT	EXACT
0.2	EXACT	Z_o = 101.6 *LB = 95.5 UB = 107.2 Est. 101.	Z_o = 71.68 LB = 70.79 UB = 72.53 Est. 71.65	Z_o = 48.54 LB = 48.50 UB = 48.61 Est. 48.55
0.5	EXACT	Z_o = 65.21 LB = 63.95 UB = 66.40 Est. 65.15	Z_o = 45.98 LB = 45.90 UB = 46.09 Est. 46.00	Z_o = 30.92 LB = 30.908 UB = 30.912 Est. 30.910

*From Ref. 1. Results from Ref. 6: LB = 98.1, UB = 103.7

Table II. Check on universal impedance formula for extreme cases. ($\kappa_e = 1$)

$\frac{\rho}{\sigma}$ =	0.01	0.05	0.1
$\frac{v}{\sigma}$ =	0.01	0.05	0.1
Z_o =	288.5	188.4	144.7
LB =	280.8	184.2	142.0
UB =	280.8	184.5	143.4

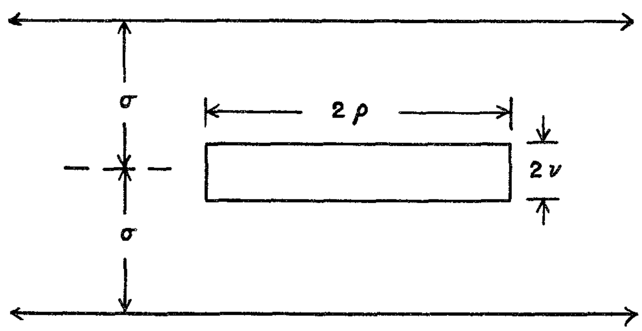

Fig. 1 - Cross-sectional view of strip transmission line with rectangular center conductor.